DEFINITE INTEGRATION

1. Let f(x) be a function defined on [a, b]. If $\int f(x)\,dx = F(x)$, then F(b) – F(a) is called the ***definite integral*** of f(x) over [a, b]. It is denoted by $\int_a^b f(x)dx$. The real number a is called the ***lower limit*** and the real number b is called the ***upper limit***.

2. $\int f(x)\,dx = F(x) + c \Rightarrow \int_a^b f(x)\,dx = F(x)]_a^b = F(b) - F(a)$

3. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

4. If f(x) is an integrable function on [a, b] and g(x) is derivable on [a, b] then
$\int_a^b (fog)(x)\,g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx$.

5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

6. If a < c < b, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

7. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

8. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

9. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f(x) is an even function;
   = 0, if f(x) is an odd function.

10. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f(2a – x) = f(x).= 0 if f(2a – x) = –f(x)

11. $\int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx$, if f(a – x) = f(x)= 0, if f(a – x) = –f(x)

12. If f(x) is a periodic function with period 'a' then $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$.

13. $\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)}\,dx = \int_0^{\pi/2} \frac{f(\cos x)}{f(\sin x) + f(\cos x)}dx = \frac{\pi}{4}$.

14. $\int_0^{\pi/2} \frac{f(\tan x)}{f(\tan x) + f(\cot x)}\,dx = \int_0^{\pi/2} \frac{f(\cot x)}{f(\tan x) + f(\cot x)}dx = \frac{\pi}{4}$.

15. $\int_0^{\pi/2} \frac{f(\sec x)}{f(\sec x)+f(\operatorname{cosec} x)}\,dx = \int_0^{\pi/2} \frac{f(\operatorname{cosec} x)}{f(\sec x)+f(\operatorname{cosec} x)}\,dx = \frac{\pi}{4}$.

16. If $I_n = \int_0^{\pi/2} \sin^n x\,dx$ then $I_n = \frac{n-1}{n} I_{n-2}$.

$\therefore \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\frac{1}{2}\cdot\frac{\pi}{2}$ if n is even $= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\frac{2}{3}.1$ if n is odd

17. If $I_n = \int_0^{\pi/2} \cos^n x\,dx$ then $I_n = \frac{n-1}{n} I_{n-2}$.

18. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

19. If $I_n = \int_0^{\pi/4} \tan^n x\,dx$ then $I_n + I_{n-2} = \frac{1}{n-1}$ and hence $I_n = \frac{1}{n-1} - \frac{1}{n-3} + \frac{1}{n-5} - \frac{1}{n-7}\ldots\ldots I_0$ or I_1 according as n is even or odd. Here $I_0 = \frac{\pi}{4}$, $I_1 = \frac{1}{2}\log 2$.

20. If $I_n = \int_0^{\pi/4} \sec^n x\,dx$ then $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$.

21. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx$ then $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$.

$\therefore I_{m,n} = \frac{(m-n)}{(m+n)}.\frac{(m-3)}{(m+n-2)}.\frac{(m-5)}{(m+n-4)}\ldots\frac{1}{n+1}.$, if n is odd

$= \frac{(m-n)}{(m+n)}.\frac{(m-3)}{(m+n-2)}.\frac{(m-5)}{(m+n-4)}\ldots\int_0^{1/2} \cos^n x.$, if n is even

22. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx$ then $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$.

23. i) $\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx = \frac{\pi}{2}$; ii) $\int_0^\infty \frac{1}{x^2+a^2}\,dx = \frac{\pi}{2a}$ iii) $\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi a^2}{4}$

24. i) $\int_0^\pi \sin mx \sin nx\,dx = \begin{cases} 0, \text{if m, n are different positive integers} \\ \pi/2, \text{if } m = n \end{cases}$

ii) $\int_0^\pi \cos mx \cos nx\,dx = \begin{cases} 0, \text{if m, n are different positive integers} \\ \pi/2, \text{if } m = n \end{cases}$

iii) $\int_0^\pi \sin mx \cos nx\,dx = \begin{cases} 0, \text{if } m \ne n \text{ and } m+n \text{ is odd} \\ \frac{2m}{m^2+n^2}, \text{if } m = n \end{cases}$

25. i) $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$ ii) $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

26. $\int_0^{\pi/2}\log\sin x\,dx = \int_0^{\pi/2}\log\cos x\,dx = -\frac{\pi}{2}\log 2$.

27. $I_n = \int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx$ If n is even then $I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}....\frac{1}{2}.\frac{\pi}{2}$

If n is odd, then $I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}....\frac{2}{3}.1$

28. $I_{m,n} = \int_0^{\pi/2}\sin^m x\cos^n x\,dx$, If m is odd, then $I_{m,n}=\frac{m-1}{m+n}.\frac{m-3}{m+n-2}...\frac{2}{n+3}.\frac{1}{n+1}$

If n is odd, then $I_{m,n}=\frac{n-1}{m+n}.\frac{n-3}{m+n-2}...\frac{2}{m+3}.\frac{1}{m+1}$

If m is even and n is even then $I_{m,n}=\frac{m-1}{m+n}.\frac{m-3}{m+n-2}...\frac{1}{n+2}.\frac{n-1}{n}...\frac{1}{2}.\frac{\pi}{2}$.

29. $\int_0^{\pi/4}[\tan^n x + \tan^{n-2} x]dx = \frac{1}{n-1}$

30. $\int_{\pi/4}^{\pi/2}[\cot^n x + \cot^{n-2} x]dx = \frac{1}{n-1}$

31. If a > b > 0, then $\int_0^{\pi/2}\frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}$

32. If 0 < a < b, then $\int_0^{\pi/2}\frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}}{\sqrt{b+a}-\sqrt{b-a}}\right|$

33. If a > b > 0, then $\int_0^{\pi/2}\frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}$

34. If 0 < a < b, then $\int_0^{\pi/2}\frac{dx}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}}{\sqrt{b+a}-\sqrt{b-a}}\right|$